Balanced Districting on Grid Graphs with Provable Compactness and Contiguity*

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Summary of Results. Given a weighted graph $G = (V, E)$ with vertex weights $w : V \to \mathbb{R}$ and a desired number of parts k, problems involving partitioning the vertex set V into k parts $V_1, \ldots, V_k$ are among the oldest and most fundamental problems in operations research. Solutions to these problems are necessary in applications ranging from drawing voting and school districts to creating police zones and clustering. The applicability of graph partitioning solutions in practice often depends on solution quality in terms of three criteria: contiguity, that each part of the partition should induce a contiguous region; balance, that the total vertex weight in each part V_i should be "close" to the average weight of any of the k parts; and compactness, that each part should not be too oblong or highly non-convex. Different formulations for partitioning problems exist, dependent on the modeling choice for each of these objectives. Most of the existing literature, however, focuses on ensuring compactness and balance, and neglects contiguity of parts.

Optimizing for balance of node weights in any part while maximizing compactness and ensuring contiguity of each part results in a multi-objective optimization problem. We convert this into a single-objective model, treating compactness (the total number of cut edges between each part) as the objective and requiring contiguity and approximate balance of each part of the partition. Our key contribution is to obtain provable compactness guarantees and computational tractability for planar graph instances while ensuring balanced, contiguous of regions. We give two approaches to produce districting plans: combinatorial algorithms for the partition problem, revisiting an old striping method introduced in 1996 by Christou and Meyer, and a simulated annealing heuristic method. Our focus is on producing a good solution efficiently in the case where vertex weights $w(v)$ are not uniform.

For applications such as police or fire department districting, one can superimpose a grid graph on a geographical region with vertex weights $w(v)$ that approximate the crime rate or workload in the corresponding square of the grid. Our modified striping method for unweighted rectangular grid graphs guarantees contiguity. It improves the previous known multiplicative approximation factor for this instance by an asymptotic factor of 6 when perimeter is used as the compactness metric, and we extend the multiplicative approximation guarantee by removing conditions on the grid size and number of parts.

In many applications, vertex weights $w(\cdot)$ are often estimates based on historical data. In practice, however, these estimates may be inaccurate, and partitioning based on these weights may result in unintended consequences. For instance, if equal police resources have been allocated to parts with different crime rates, reported crime rates may be disproportionately higher in the parts with a lower true crime rate. To model these issues, we also consider a stochastic model where vertex weights are estimates based on historical data. We show guarantees on the expected balance of partitions in this model and combine ideas from stochastic load balancing with combinatorial striping methods to further improve expected balance.

We compare our methods on several real-world and synthetic instances, including redistricting problems for the fire department and on police beats in South Fulton City, Georgia. To get tractable solutions in practice, we combine the striping approaches with simulated annealing to balance weights while maintaining contiguity and compactness. Our case studies exemplify the importance of scalable approaches to larger geographic regions, and we show significant improvements in workload balance.

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